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LOCALIZED FAMILIES OF BENDING WAVES IN A THIN MEDIUM-LENGTH CYLINDRICAL SHELL UNDER PRESSURE

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The initial-boundary-value problem for the equations describing motion of a thin, medium-length, non-circular cylindrical shell is examined. The shell edges are not necessarily plane curves, with the conditions of a joint support, a rigid clamp or a free edge being considered as the boundary conditions. The shell is supposed to experience normal internal (or external) dynamic pressure which may be non-uniform in the circumferential direction. It is assumed that the initial displacements and velocities of the points at the shell middle surface are functions decreasing rapidly away from some generatrix. Using the complex WKB method the asymptotic solution of the governing equations is constructed by superimposing localized families (wave packets) of bending waves running in the circumferential direction. The dependence of frequencies, group velocities, amplitudes and other dynamic characteristics upon variable pressure and geometrical parameters of the shell are studied. As an example, the wave forms of motion of a circular cylindrical shell with sloping edges under growing dynamic pressure are considered. The effect of localization of bending vibrations near the longest generator as well as the effects of reflection, focusing and increasing amplitude in the running wave packets are revealed.

1. INTRODUCTION

Thin cylindrical shells or panels are used as elements of high-speed airborne/spaceborne vehicles, underwater objects and many other engineering structures experiencing dynamic loading. Non-stationary vibrations running in the circumferential direction in thin medium-length cylinders caused by the transient dynamic forces and/or the initial conditions (displacements and velocities) on the shell surface are especially complicated for an analysis, being at the same time of great practical interest [1–7]. In a number of cases, numerous numerical methods do not reveal any mechanical effects which are inherent to the transitional wave processes in thin shells, and known analytical approaches are found productive only after introducing some simplifying suppositions. For example, replacing a medium-length cylindrical shell by an infinitely long one and assuming the independence of loading on the axial co-ordinate were made in references [1–5]. In such a statement the initial two-dimensional problem (with respect to curvilinear co-ordinates) is reduced to a one-dimensional problem, a shell being actually replaced by a ring. The method of investigation of the simplified problem depends upon both the character of loading and the initial conditions. So, to examine non-stationary vibrations of a thin elastic infinitely long cylindrical shell (i.e., a ring) to an arbitrary distribution of the initial radial velocities, McIvor

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[2] has utilized the method of an expansion of the desired displacements into a series of ring eigenmodes. And in reference [5], the dynamic response of an infinite cylindrical shell subjected to the action of a plane shock wave has been analyzed only in a small initial segment of time.

Constructions of explicit non-stationary solutions for a medium-length cylindrical shell taking into account the boundary conditions have been carried out in references [6, 7]. In particular, in reference [7], based on the Flügge shell theory (including the effects of shear deformation and rotatory inertia), an exact formal solution has been presented for the dynamic response of a freely supported circular cylindrical shell of finite length subjected to time-dependent, arbitrarily distributed surface loading, the solution has being found by means of an expansion of the shell displacement into a series in eigenfunctions of the corresponding boundary-value problems. However, these and many other papers deal with cylindrical shells having constant geometrical parameters, whereas non-stationary vibrations of thin, non-circular cylinders and panels with arbitrary (not plane or oblique) edges are still insufficiently investigated.

The peculiarity of the shells with variable geometrical characteristics lies in the possible localization of vibrations in the so-called "weakest" spots on the shell surface. For instance, the presence of an oblique edge in a circular cylindrical shell may be the cause of the localization of free low-frequency bending vibrations in a vicinity of the longest generator which will be the weakest one [8]. It is evident that the nature of the transient bending forms of vibrations of a cylindrical shell having the weakest line may in essence differ from the non-stationary motion of a shell with constant parameters. An attempt to study the influence of the oblique edge on transient bending vibrations of a thin medium-length cylinder has been undertaken in reference [9], where by using the new asymptotic approach [10], solutions of the governing equations have been found in the form of packets of short bending waves running in the circumferential direction. A thorough analysis of the constructed solutions has allowed one to detect a series of new mechanical effects such as the reflection of some packets from a sufficiently short generator of a shell, and focusing of the travelling packets being accompanied by increasing wave amplitudes. Afterwards, this method has been applied to studying the packets of bending, longitudinal and torsional waves running in the axial direction in an infinitely long cylindrical shell including the effect of initial tensions due to non-uniform static internal pressure [11].

In this paper, the approach developed in references [9–11] is applied to examine non-stationary localized bending vibrations of a thin, medium-length, non-circular cylindrical shell with arbitrary edges under time-dependent, arbitrarily distributed over the circumferential co-ordinate, normal pressure. The shell is supposed to have local perturbations which are treated as the initial conditions in its surface. Significant attention in this investigation is paid to localized wave processes in a neighborhood of the weakest generator. As a special case, the solution constructed in the paper permits one to describe low-frequency free vibrations near the weakest line [8].

2. SETTING THE PROBLEM

Consider an elastic thin non-circular medium length cylindrical shell of thickness h. Let ρ be the density, E be Young's modulus, and ν be Poisson ratio of the material. A co-ordinate system as illustrated in Figure 1 is chosen in such a way that $d\sigma^2 = R^2(ds^2 + d\varphi^2)$ is the first quadratic form of the middle shell surface. The radius of curvature is $R_2 = R/k(\varphi)$. Here R is the characteristic dimension of the shell which may be

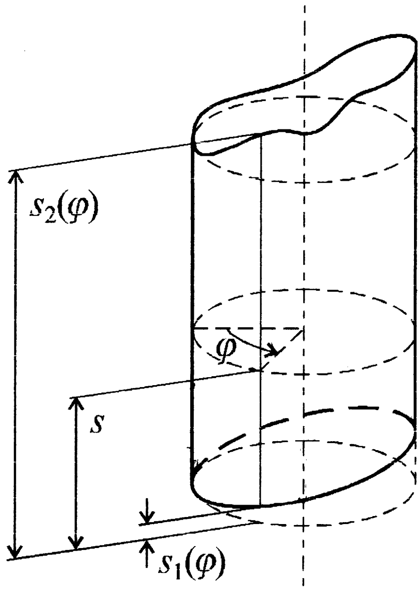

Figure 1. The neutral surface of the thin cylindrical shell with non-plane edges and the co-ordinate system.

introduced as the maximum value of R_2. Suppose that the shell is bounded by the two not necessarily plane edges

$$s_1(\varphi) \leqslant s \leqslant s_2(\varphi) \tag{2.1}$$

and may be non-closed in the direction of φ (the case of a non-circular cylindrical panel), with $-\pi < \varphi_1 \leqslant \varphi \leqslant \varphi_2 \leqslant \pi$. Let the shell be under the non-uniform, dynamic load $\mathbf{Q}^* = (Q_1^*, Q_2^*, Q_3^*)$, where Q_1^*, Q_2^* and Q_3^* are the tangential and normal components of the force $\mathbf{Q}^*$ applied to the shell surface. It is assumed that Q_j^* ($j = 1, 2, 3$) are slowly varying functions with respect to both space co-ordinates and time so that the dynamic stress state of the shell due to the load $\mathbf{Q}^*$ may be specified only by the axial, hoop and shear stresses.

$$T_1^* = -Eh\varepsilon^6 T_1(s, \varphi, t), \quad T_2^* = -Eh\varepsilon^6 T_2(s, \varphi, t), \quad T_3^* = -Eh\varepsilon^6 T_3(s, \varphi, t), \tag{2.2}$$

respectively, which are easily found from the equations of the membrane shell theory [12]. In equations (2.2), $\varepsilon^8 = h^2/[12R^2(1-\nu^2)]$ is a natural small parameter characterizing the shell thinness, T_i are the dimensionless stresses, $t = t^*/t_c^*$ is dimensionless time, and $t_c^* = \sqrt{R^2\rho/(E\varepsilon^6)}$ is the characteristic time. The case when the hoop stress $T_2(\varphi, t)$ does not depend on s is considered here (although this requirement is not obligatory, it simplifies only the following asymptotic constructions). In addition, it is assumed that all the functions $k(\varphi)$, $s_i(\varphi)$, $T_1(s, \varphi, t)$, $T_2(\varphi, t)$, $T_3(s, \varphi, t)$ are infinitely differentiable ones with respect to φ, and $T_1(s, \varphi, t)$, $T_3(s, \varphi, t)$ are twice differentiable with respect to s and t so that

$$k,\ s_i, \quad \partial^m k/\partial\varphi^m, \quad \partial^m s_i/\partial\varphi^m = O(1), \tag{2.3a}$$

$$\partial^m T_j/\partial\varphi^m, \quad \partial^n T_1/\partial s^n, \quad \partial^n T_1/\partial t^n, \quad \partial^n T_3/\partial s^n, \quad \partial^n T_3/\partial t^n = O(1) \tag{2.3b}$$

where $m = 1, 2, \ldots,\; i, n = 1, 2\;\; j = 1, 2, 3$. The definition of the symbol O is given in Appendix A (see also references [13–15]). Further restrictions to the functions T_j will be introduced below.

The asymptotic correlations (2.3b) emphasize the slow variability of the stresses T_j over s, φ and t. Then, for analysis of the short waves running in the circumferential direction the following system of equations [16], including the effect of the initial stresses caused by the load $\mathbf{Q}^*$, and written in dimensionless form, may be used:

$$\varepsilon^4 \Delta^2 W + k(\varphi)\partial^2 \Phi/\partial^2 s + \varepsilon^2 \Delta_T W + \varepsilon^2 \partial^2 W/\partial t^2 = 0,$$

$$\varepsilon^4 \Delta^2 \Phi - k(\varphi)\partial^2 W/\partial s^2 = 0. \tag{2.4}$$

Here

$$\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \varphi^2}, \quad \Delta_T W = \frac{\partial}{\partial \varphi}\left(T_2 \frac{\partial W}{\partial \varphi}\right) + \frac{\partial}{\partial s}\left(T_3 \frac{\partial W}{\partial \varphi}\right) + \frac{\partial}{\partial \varphi}\left(T_3 \frac{\partial W}{\partial s}\right) + \frac{\partial}{\partial s}\left(T_1 \frac{\partial W}{\partial s}\right) \tag{2.5}$$

and the dimensionless magnitudes are introduced as follows:

$$W = W^*/R, \quad \Phi = \Phi^*/(\varepsilon^4 R^2 h E),$$

where W^* is the normal deflection, Φ^* is the stress function. Equations (2.4) represent the dynamic stress state of the shell perturbed from its membrane stress state, and they were used by many researchers for studying dynamic instability of thin cylindrical shells under stresses T_j being periodic functions of time (see, e.g., references [17, 18]).

Consider that on each of the shell edges $s = s_1(\varphi)$, $s = s_2(\varphi)$ there are one of the three groups of the boundary conditions, namely, the joint support group, the rigid clamp group and the group of a free edge. Each of the first two groups includes six versions of the boundary conditions, and the last group for a free edge contains two versions. All the mentioned boundary condition variants are listed in a book [19]. The dynamic stress state of the shell consists of the basic dynamic stress state and the dynamic edge-effect integrals describing the shell behavior in some small neighborhood of each edge [12, 20]. To study the basic state on each edge, one only needs to satisfy two basic conditions [12]. Apart from terms of the order ε^2 these conditions have the form [8, 19]

$$W = \partial^2 W/\partial s^2 = 0 \quad \text{at } s = s_i(\varphi), \tag{2.6a}$$

$$W = \partial W/\partial s = 0 \quad \text{at } s = s_i(\varphi), \tag{2.6b}$$

$$\Phi = \partial \Phi/\partial s = 0 \quad \text{at } s = s_i(\varphi), \tag{2.6c}$$

for the joint supported, rigid clamped and free edges respectively. Combinations of conditions (2.6a)–(2.6c) on the edges $s = s_1(\varphi)$, $s_2(\varphi)$ may be considered as well.

The wave forms of motion caused by the initial displacements and velocities are

$$W|_{t=0} = W^{\circ}(s, \varphi, \varepsilon)\exp[\mathrm{i}\varepsilon^{-1}S^{\circ}(\varphi)],$$

$$\dot{W}|_{t=0} = \mathrm{i}\varepsilon^{-1}V^{\circ}(s, \varphi, \varepsilon)\exp[\mathrm{i}\varepsilon^{-1}S^{\circ}(\varphi)], \tag{2.7}$$

where

$$\mathrm{i} = \sqrt{-1}, \quad S^{\mathrm{o}}(\varphi) = a^{\mathrm{o}}\varphi + \tfrac{1}{2}b^{\mathrm{o}}\varphi^2, \quad a^{\mathrm{o}} > 0, \quad \mathrm{Im}\, b^{\mathrm{o}} > 0, \tag{2.8}$$

$$a^{\mathrm{o}}, |b^{\mathrm{o}}|, |W^{\mathrm{o}}|, |V^{\mathrm{o}}|, |\partial W^{\mathrm{o}}/\partial s|, |\partial V^{\mathrm{o}}/\partial s| = O(1) \text{ when } \varepsilon \to 0 \tag{2.9}$$

will be studied below. In equations (2.7), (2.8), a^{o}, b^{o} are real and complex constants, respectively, and W^{o}, V^{o} are complex-valued functions satisfying one version of the boundary conditions (2.6).

The real and imaginary parts of functions (2.7), with account taken of the last inequality in equations (2.8), define the two initial wave packets localized near the line $\varphi = 0$. Functions (2.7) approximate perturbations which may be generated in the shell by some transient forces applied along the line $\varphi = 0$. The wave packets like those (2.7) may also appear [21, 22] as a result of the parametric excitation of the cylindrical shell having variable geometric parameters (e.g., a curvature, a thickness or a generatrix length) in the circumferential direction, or experiencing non-uniform load. It is known that the non-homogeneity of the geometric parameters mentioned above may cause the localization of modes of low-frequency free vibrations of a shell in a vicinity of the so-called weakest generatrix on the shell surface [8]. It is this circumstance that can lead to the localization of parametric vibrations near the weakest line as well. So, local free vibrations and parametric instability of non-circular cylindrical shells under static and additional periodic axial loads that are not uniform in the circumferential direction have been analyzed in references [21, 22], where solutions of the governing equations have been constructed in the form of functions analogous to equations (2.7).

3. THE APPROACH

In the case when the stresses T_j are absent, the initial-boundary-value problem (2.4), (2.6), (2.7) has already been considered [9, 23]. If the shell edges lie in planes perpendicular to the generatrix, i.e., s_1, s_2, are constants, the solution of this problem is easily found [23] as an expansion in beam functions along the generatrix, with the expansion permitting the original initial-boundary-value problem (2.4), (2.6), (2.7) to be reduced to an initial problem, the solution of which can be constructed by Maslov's method [24]. However, this method is ineligible for shells with sloping edges. Quite a different approach has been undertaken in reference [9] for studying the packets of bending waves running in the circumferential direction in a medium-length cylindrical shell with slanting edges. The basic concepts of this new method lies in introducing the center of the wave packet and a local co-ordinate system connected with this center [10]. It is this approach that is to be used now to construct the asymptotic solution of problem (2.4), (2.6), (2.7). Observing incidentally, this method has been used earlier to analyze the packets of bending, longitudinal and torsional waves travelling in the axial direction in an infinite shell of revolution [25], and in an infinite cylindrical shell under non-uniform internal pressure as well [11].

Let $y_1(s, \varphi), \ y_2(s, \varphi), \ldots, y_n(s, \varphi), \ldots$ be an infinite orthonormalized system of eigenfunctions of the equation

$$\mathrm{d}^4 y/\mathrm{d}s^4 - \lambda y = 0 \tag{3.1}$$

with one of the three variants of the boundary conditions

$$y = \mathrm{d}^2 y/\mathrm{d}s^2 = 0 \quad \text{for } s = s_1(\varphi), \ s_2(\varphi), \tag{3.2a}$$

$$y = \mathrm{d}y/\mathrm{d}s = 0 \quad \text{for } s = s_1(\varphi),\ s_2(\varphi), \tag{3.2b}$$

$$\mathrm{d}^2y/\mathrm{d}s^2 = \mathrm{d}^3y/\mathrm{d}s^3 = 0 \quad \text{for } s = s_1(\varphi),\ s_2(\varphi) \tag{3.2c}$$

and $\lambda_1(\varphi), \lambda_2(\varphi), \dots, \lambda_n(\varphi), \dots$ be a corresponding sequence of eigenvalues. For instance, equation (3.1), with conditions (3.2a) in mind, describes free bending vibrations of a joint supported beam of the length $l(\varphi) = s_2(\varphi) - s_1(\varphi)$. For definiteness, the case of a joint support (3.2a) on the both edges will be examined here, with

$$y_n(s, \varphi) = \sin\{\pi n[s - s_1(\varphi)]/l(\varphi)\}, \quad \lambda_n(\varphi) = [\pi n/l(\varphi)]^4 \tag{3.3}$$

although all further calculations will also be valid for other boundary conditions.

Suppose that the functions $W^{\mathrm{o}}(s, \varphi, \varepsilon)$, $V^{\mathrm{o}}(s, \varphi, \varepsilon)$ appearing in initial conditions (2.7) satisfy equations (2.6a). Then for any $\varphi \in [\varphi_1, \varphi_2]$, the functions W^{o}, V^{o} can be expanded [26] in terms of the eigenfunctions $y_n(s, \varphi)$ into uniformly convergent series in the section $[\varphi_1, \varphi_2]$

$$W^{\mathrm{o}} = \sum_{n=1}^{\infty} w_n^{\mathrm{o}}(\varphi, \varepsilon) y_n(s, \varphi), \quad w_n^{\mathrm{o}} = \int_{s_1(\varphi)}^{s_2(\varphi)} W^{\mathrm{o}}(s, \varphi, \varepsilon) y_n(s, \varphi)\,\mathrm{d}s,$$

$$V^{\mathrm{o}} = \sum_{n=1}^{\infty} v_n^{\mathrm{o}}(\varphi, \varepsilon) y_n(s, \varphi), \quad v_n^{\mathrm{o}} = \int_{s_1(\varphi)}^{s_2(\varphi)} V^{\mathrm{o}}(s, \varphi, \varepsilon) y_n(s, \varphi)\,\mathrm{d}s, \tag{3.4}$$

Let w_n^{o}, v_n^{o} be polynomials of $\varepsilon^{-1/2}\varphi$ whose coefficients are regular functions of ε. This assumption involves the presence of a finite number of oscillations in the amplitude of the initial wave packet. Then the functions w_n^{o}, v_n^{o} may be represented by the series

$$w_n^{\mathrm{o}} = \sum_{m=0}^{\infty} \varepsilon^{m/2} w_{nm}^{\mathrm{o}}(\zeta), \quad w_{nm}^{\mathrm{o}} = \sum_{k=0}^{M_{nm}} c_{nmk}^{\mathrm{o}} \zeta^k,$$

$$v_n^{\mathrm{o}} = \sum_{m=0}^{\infty} \varepsilon^{m/2} v_{nm}^{\mathrm{o}}(\zeta), \quad v_{nm}^{\mathrm{o}} = \sum_{k=0}^{M_{nm}} d_{nmk}^{\mathrm{o}} \zeta^k, \tag{3.5}$$

where $\zeta = \varepsilon^{-1/2}\varphi$, and w_{nm}^{o}, v_{nm}^{o} are polynomials of degree M_{nm} with complex coefficients c_{nmk}^{o}, d_{nmk}^{o} such that $|c_{nmk}^{\mathrm{o}}|, |d_{nmk}^{\mathrm{o}}| = O(1)$.

Upon taking into account the linearity of equations (2.4) as well as expansions (3.4), the solutions of the boundary-value problem (2.4), (2.6a), (2.7) may be presented in the form [9]:

$$W(s, \varphi, t, \varepsilon) = \sum_{n=1}^{\infty} \tilde{W}_n(s, \varphi, t, \varepsilon), \quad \Phi(s, \varphi, t, \varepsilon) = \sum_{n=1}^{\infty} \tilde{\Phi}_n(s, \varphi, t, \varepsilon), \tag{3.6}$$

where $\tilde{W}_n$, $\tilde{\Phi}_n$ $(n = 1, 2, \dots)$ are the required functions localized in a neighborhood of a generatrix $\varphi = q_n(t)$. Here $q_n(t)$ is a twice differentiable function such that

$$q_n(0) = 0. \tag{3.7}$$

The pair $\tilde{W}_n$, $\tilde{\Phi}_n$ will be called the nth wave packet with the center at $\varphi = q_n(t)$.

Now hold the number n fixed and study the behavior of the nth wave packet. In view of the local character of the functions $\tilde{W}_n$, $\tilde{\Phi}_n$, it is convenient to go over to a local co-ordinate system [10]:

$$\varphi = q_n(t) + \varepsilon^{1/2}\xi_n. \tag{3.8}$$

Here parameters ξ_n, s define the position of a point on the shell surface with respect to the moving center $q_n(t)$. In the new co-ordinate system equations (2.4) can be rewritten as

$$\varepsilon^2 \frac{\partial^4 \tilde{W}_n}{\partial \xi_n^4} + 2\varepsilon^3 \frac{\partial^4 \tilde{W}_n}{\partial \xi_n^2 \partial s^2} + \varepsilon^4 \frac{\partial^4 \tilde{W}_n}{\partial s^4} + k \frac{\partial^4 \tilde{\Phi}_n}{\partial s^2}$$

$$+ \varepsilon \frac{\partial}{\partial \xi_n}\left(T_2 \frac{\partial \tilde{W}_n}{\partial \xi_n}\right) + \varepsilon^{3/2} \frac{\partial}{\partial s}\left(T_3 \frac{\partial \tilde{W}_n}{\partial \xi_n}\right) + \varepsilon^{3/2} \frac{\partial}{\partial \xi_n}\left(T_3 \frac{\partial \tilde{W}_n}{\partial s}\right) + \varepsilon^2 \frac{\partial}{\partial s}\left(T_1 \frac{\partial \tilde{W}_n}{\partial s}\right)$$

$$+ \varepsilon^2 \frac{\partial^2 \tilde{W}_n}{\partial t^2} - \varepsilon^{3/2} \dot{q}_n \frac{\partial^2 \tilde{W}_n}{\partial \xi_n \partial t} + \varepsilon \dot{q}_n^2 \frac{\partial^2 \tilde{W}_n}{\partial \xi_n^2} - \varepsilon^{3/2} \ddot{q}_n \frac{\partial \tilde{W}_n}{\partial \xi_n} = 0,$$

$$\varepsilon^2 \frac{\partial^4 \tilde{\Phi}_n}{\partial \xi_n^4} + 2\varepsilon^3 \frac{\partial^4 \tilde{\Phi}_n}{\partial \xi_n^2 \partial s^2} + \varepsilon^4 \frac{\partial^4 \tilde{\Phi}_n}{\partial s^4} - k \frac{\partial^2 \tilde{W}_n}{\partial s^2} = 0, \tag{3.9}$$

where the dots $(\cdot)$ denote differentiation with respect to t.

Upon taking into account expansions (3.4), the initial conditions for the nth wave packet take the form

$$\tilde{W}_n|_{t=0} = w_n^{\mathrm{o}}(\varphi, \varepsilon) y_n(s, \varphi) \exp[\mathrm{i}\varepsilon^{-1} S^{\mathrm{o}}(\varphi)],$$

$$\dot{\tilde{W}}_n|_{t=0} = \mathrm{i}\varepsilon^{-1} v_n^{\mathrm{o}}(\varphi, \varepsilon) y_n(s, \varphi) \exp[\mathrm{i}\varepsilon^{-1} S^{\mathrm{o}}(\varphi)] \tag{3.10}$$

and the boundary conditions are

$$\tilde{W}_n = \partial^2 \tilde{W}_n / \partial s^2 = 0 \tag{3.11}$$

for the joint supported shell.

To avoid inconvenience the subscript n is omitted in what follows. For example, the notations w_{nm}^{o}, M_{nm}, c_{nmk}^{o}, $\tilde{W}_n$, q_n, λ_n, y_n, ξ_n are replaced by w_n^{o}, M_m, c_{mk}^{o}, $\tilde{W}$, q, λ, y, ξ respectively.

The functions $k(\varphi)$, $s_i(\varphi)$, $T_j(\varphi, t)$, $y(s, \varphi)$ are expanded into a series in a neighborhood of the center $q(t)$. In particular,

$$T_2(\varphi, t) = T_2[q(t), t] + \varepsilon^{1/2} T_2'[q(t), t]\xi + \tfrac{1}{2}\varepsilon T_2''[q(t), t]\xi^2 + \cdots$$

where the prime $(')$ means differentiation with respect to φ.

Following references [9, 10] the formal asymptotic solution of the initial-boundary-value problem (3.9)–(3.11) is assumed to be of the form

$$\tilde{W} \cong \sum_{m=0}^{\infty} m^{m/2} w_m(s, \xi, t) \exp[\mathrm{i}\,\varepsilon^{-1} S(\xi, t, \varepsilon)],$$

$$\tilde{\Phi} \cong \sum_{m=0}^{\infty} m^{m/2} f_m(s, \xi, t) \exp[\mathrm{i}\,\varepsilon^{-1} S(\xi, t, \varepsilon)], \tag{3.12a}$$

$$S = \int_0^t \omega(\tau)\,\mathrm{d}\tau + \varepsilon^{1/2} p(t)\xi + \frac{1}{2}\varepsilon b(t)\xi^2, \tag{3.12b}$$

$$\operatorname{Im} b(t) > 0 \quad \text{for any } t \geqslant 0, \tag{3.13}$$

where the symbol $\cong$ means that the series is an *asymptotic expansion* of the function $\tilde{W}$ or $\tilde{\Phi}$ in the *Poincaré sense* (see the definition in Appendix A), $w_m(s, \xi, t)$, $f_m(s, \xi, t)$ are polynomials in ξ with complex coefficients being functions of t and s, $|\omega(t)|$ is the momentary frequency of vibrations of the shell in a neighborhood of the center $\varphi = q(t)$, $p(t)$ is the wave number determining the variability of waves in the circumferential direction, and the function $b_n(t)$ characterizes the width of the nth wave packet, inequality (3.13) guaranteeing attenuation of wave amplitudes within the packet.

It should be emphasized that solutions in the form (3.12), when $q = 0$, and ω, p, b are constants, have been constructed earlier in the problems on the local buckling and vibration of thin medium-length cylindrical shells near the "weakest" generator [8, 19]. Solutions of this type are called the WKB approximations. This name comes from the first letters of the author's names: Wentzel, Kramers and Brillouin, who first proposed such asymptotic construction in problems of quantum mechanics. The history and underlying concepts of the WKB method and its non-stationary variants are briefly presented in reference [11].

Substituting expansions (3.12) into equations (3.9), equating the coefficients of like powers of $\varepsilon^{m/2}$ to zero, and eliminating f_m, one obtains the sequence of differential equations

$$\sum_{j=0}^{m} L_j w_{m-j} = 0, \quad m = 0, 1, 2, \ldots, \tag{3.14}$$

where

$$L_0 = \frac{k^2(q(t)]}{p^4(t)}\frac{\partial^4}{\partial s^4} + p^4(t) - T_2[q(t), t]p^2(t) - [\omega(t) - \dot{q}(t)p(t)]^2,$$

$$L_1 = (bL_p + L_q + \dot{p}L_\omega)\xi - \mathrm{i}L_p\partial/\partial\xi,$$

$$L_2 = (b^2L_{pp} + 2bL_{pq} + L_{qq} + \dot{p}^2L_{\omega\omega} + 2\dot{p}L_{\omega q} + 2\dot{p}bL_{\omega p} + \dot{b}L_\omega)\xi^2$$

$$-\frac{1}{2}L_{pp}\frac{\partial^2}{\partial\xi^2} - \mathrm{i}(bL_{pp} + L_{pq} + \dot{p}L_{\omega p})\xi\frac{\partial}{\partial\xi} - \mathrm{i}L_\omega\frac{\partial}{\partial t}$$

$$-\mathrm{i}\left(\frac{1}{2}bL_{pp} + \frac{1}{2}\dot{\omega}L_{\omega\omega} + \dot{p}L_{\omega p} + \ddot{q}p + N\right), \ldots,$$

$$N = -\frac{4k[q(t)]k'[q(t)]}{p^5(t)}\frac{\partial^4}{\partial s^4} - 2p(t)\left\{T_3[s, q(t), t]\frac{\partial}{\partial s} - \frac{\partial T_3}{\partial s}[s, q(t), t]\right\}. \tag{3.15}$$

The subscripts p, q, ω in equations (3.15) denote differentiation with respect to the corresponding variables p, q ω. Operators L_m for $m \geqslant 3$ are not written out here in the explicit form because of its awkwardness. Note only that the axial stress T_1 is contained in the operator L_3.

The functions f_m are found one after another from the inhomogeneous equations and can be expressed in terms of $w_0, w_1, \ldots, w_m$. In particular,

$$f_0 = \frac{k(q)}{p^4}\frac{\partial^2 w_0}{\partial s^2}, \quad f_1 = \frac{k(q)}{p^4}\frac{\partial^2 w_1}{\partial s^2} + \frac{4\mathrm{i}k(q)}{p^5}\frac{\partial^3 w_0}{\partial s^2 \partial \xi}$$

$$- \frac{4k(q)b}{p^5}\xi\frac{\partial^2 w_0}{\partial s^2} + \frac{k'(q)}{p^4}\xi\frac{\partial^2 w_0}{\partial s^2}.$$

Substituting equations (3.12) into equations (3.11) produces the sequence of boundary conditions for w_m:

$$w_0 = 0, \quad \frac{\partial^2 w_0}{\partial s^2} = 0, \tag{3.16a}$$

$$w_1 + \xi s_i' \frac{\partial w_0}{\partial s} = 0, \quad \frac{\partial^2 w_1}{\partial s^2} + \xi s_i' \frac{\partial^3 w_0}{\partial s^3} = 0, \tag{3.16b}$$

$$w_2 + \xi s_i' \frac{\partial w_1}{\partial s} + \frac{1}{2}\xi^2\left(s_i'' \frac{\partial w_0}{\partial s} + s_i'^2 \frac{\partial^3 w_0}{\partial s^2}\right) = 0,$$

$$\frac{\partial^2 w_2}{\partial s^2} + \xi s_i' \frac{\partial^3 w_1}{\partial s^3} + \frac{1}{2}\xi^2\left(s_i'' \frac{\partial^3 w_0}{\partial s^3} + s_i'^2 \frac{\partial^4 w_0}{\partial s^4}\right) - \frac{4\mathrm{i}s_i'}{p}\frac{\partial^3 w_0}{\partial s^3} = 0, \ldots \tag{3.16c}$$

at $s = s_i[q(t)]$, $i = 1, 2$.

Note that equations (3.16) guarantee a realization of the boundary conditions (3.11) merely in some small vicinity of the center $\varphi = q(t)$. However, there is no sense to satisfy conditions (3.11) on the whole segment $\varphi_1 \leqslant \varphi \leqslant \varphi_2$ because of the exponential decay of the wave amplitude far from the line $\varphi = q(t)$.

The sequence of one-dimensional boundary-value problems (3.14), (3.16) is used for the determination of unknown functions $q(t)$, $\omega(t)$, $p(t)$, $b(t)$, $w_m(t)$, $f_m(t)$. Consider these problems step-by-step for $m = 0, 1, 2, \ldots$

3.1. ZEROTH ORDER APPROXIMATION

In the zeroth order approximation ($m = 0$), one has the homogeneous ordinary differential equation

$$L_0 w_0 = 0 \tag{3.17}$$

with the boundary conditions (3.16a). The solution of the boundary-value problem (3.17), (3.16a) may be presented in the form

$$w_0 = P_0(\xi, t) y[s, q(t)], \tag{3.18}$$

where $P_0(\xi, t)$ is an unknown polynomial in ξ with coefficients being smooth functions of time t. Substituting equation (3.18) into equation (3.17) yields the relation

$$\omega(t) = \dot{q}(t)p(t) - H^{\pm}[p(t), q(t)], \tag{3.19}$$

where

$$H^{\pm}(p, q, t) = \pm \sqrt{p^4 + \frac{\lambda(q)k^2(q)}{p^4} - T_2(q, t)p^2} \tag{3.20}$$

are Hamilton functions. The signs ($\pm$) in equations (3.19) indicate the availability of two branches (positive and negative) of the solutions corresponding to the functions $H^{\pm}$. These signs are omitted in what follows, and all further constructions will be fulfilled for the function H^{+}. In equation (3.19), the first term defines the frequency of passage of the wave crests with respect to an "immovable observer" on the shell surface, and the Hamiltonian H is the instantaneous frequency of vibrations of the middle surface points within the running packets in relation to the distorted shell surface.

The polynomial $P_0(\xi, t)$ remains unknown here.

3.2. FIRST ORDER APPROXIMATION

In the first order approximation ($m = 1$), one has the non-homogeneous differential equation

$$L_0 w_1 = - L_1 w_0 \tag{3.21}$$

with the non-homogeneous boundary conditions (3.16b). Solution of the boundary-value problem (3.21), (3.16b) is presented in the form

$$w_1 = P_1(\xi, t) y[s, q(t)] + w_1^{(p)}(s, \xi, t), \tag{3.22}$$

where $P_1(\xi, t)$ is a new unknown polynomial in ξ, and $w_1^{(p)}(s, \xi, t)$ is some partial solution of equation (3.21). Upon taking into account the self-conjugacy of the boundary-value problem (3.17), (3.16a), the equality

$$\int_{s_1[q(t)]}^{s_2[q(t)]} y(L_0 w_1 + L_1 P_0 y)\,\mathrm{d}s = 0 \tag{3.23}$$

serves as the condition for the existence of w_{n1} in form (3.22). To calculate the second integrand term in equation (3.23) it is necessary to define the operators L_p, L_q (see equations (3.15)). To do this, the homogeneous boundary-value problem (3.17), (3.16a) should be differentiated over the parameters p and q:

$$L_0 w_p + L_p w_0 + 2H(\dot{q} - H_p) w_0 = 0,$$

$$w_p = 0, \quad \frac{\partial^2 w_p}{\partial s^2} = 0 \quad \text{for } s = s_i[q(t)],$$

$$L_0 w_q + L_q w_0 - 2H H_q w_0 = 0,$$

$$w_q + s_i' \frac{\partial w_0}{\partial s} = 0, \quad \frac{\partial^2 w_q}{\partial s^2} + s_i' \frac{\partial^3 w_0}{\partial s^3} = 0 \quad \text{for } s = s_i[q(t)]. \tag{3.24}$$

Equations (3.24), together with equation (3.18) imply

$$L_p w_0 = -2H(\dot{q} - H_p)P_0 y, \tag{3.25a}$$

$$L_q w_0 = 2HH_q P_0 y - L_0 P_0 y_q. \tag{3.25b}$$

By using the last equations and also the boundary conditions (3.24) for w_q (here $w_p \equiv 0$), equation (3.23) can be rewritten as the differential one

$$b(\dot{q} - H_p)\xi P_0 + (\dot{p} + H_q)\xi P_0 - \mathrm{i}(\dot{q} - H_p)\frac{\partial P_0}{\partial \xi} = 0 \tag{3.26}$$

with respect to P_{n0}. In this equation, the expressions given in the parentheses are real, and by the assumption made above, $\operatorname{Im} b_n(t) > 0$ for any $t > 0$. Hence, P_{n0} is the polynomial in ξ_n, if the functions p_n, q_n satisfy the Hamiltonian system

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q. \tag{3.27}$$

Comparison of equations (2.8), (3.10) and (3.12), with equation (3.7) in mind, gives the initial conditions

$$p(0) = a^\circ, \quad q(0) = 0 \tag{3.28}$$

for system (3.27). Let $p(t)$, $q(t)$ be a solution of system (3.27) with initial conditions (3.28). Then equation (3.25a) implies $L_p w_p = 0$, and the operator L_{n1} is simplified:

$$L_1 = (L_q + \dot{p}L_\omega)\xi. \tag{3.29}$$

Hence, the partial solution of the non-homogeneous differential equation (3.21) has the form

$$w_1^{(p)} = \xi P_0 \partial y/\partial q. \tag{3.30}$$

In this approximation, the polynomials P_0, P_1 remain undefined.

3.3. SECOND ORDER APPROXIMATION

In the second order approximation ($m = 2$), the non-homogeneous boundary-value problem (3.14), (3.16c) arises again. The compatibility condition for this problem may be deduced from the equation

$$\int_{s_1[q(t)]}^{s_2[q(t)]} y[L_0 w_2 + L_1(P_1 y + \xi P_0 y_q) + L_2 P_0 y]\,\mathrm{d}s = 0. \tag{3.31}$$

To define operators L_{pp}, L_{pq}, L_{qq} being the part of L_2, it is necessary to differentiate the boundary-value problem (3.24) with respect to the parameters p and q once more. For instance,

$$L_0 w_{qq} + 2L_q w_q - 2L_\omega H_q w_q + L_{qq} w_0 - 2L_{\omega q} H_q w_0 + L_{\omega\omega} H_q^2 w_0 - L_\omega H_{qq} w_0 = 0,$$

$$w_{qq} + 2s_i' \frac{\partial w_q}{\partial s} + s_i'' \frac{\partial w_0}{\partial s} + s_i'^2 \frac{\partial^2 w_0}{\partial s^2} = 0 \quad \text{for } s = s_i[q(t)],$$

$$\frac{\partial^2 w_{qq}}{\partial s^2} + 2s_i' \frac{\partial^3 w_q}{\partial s^2} + s_i'' \frac{\partial^3 w_0}{\partial s^3} + s_i'^2 \frac{\partial^4 w_0}{\partial s^4} = 0 \quad \text{for } s = s_i[q(t)]. \tag{3.32}$$

Inserting $L_{pp}w_0$, $L_{pq}w_0$, $L_{qq}w_0$ into equation (3.31), taking into account the boundary conditions (3.32) for w_{qq} and the identities $L_p = 0$, $w_p = 0$, $w_{pq} = 0$, $w_{pp} = 0$, one obtains the differential equation

$$(\xi^2 D_b - 2D_{\xi t})P_0 = 0 \tag{3.33}$$

with respect to P_0. Here

$$D_b = \dot{b} + H_{pp}b^2 + 2H_{pq}b + H_{qq},$$

$$D_{\xi t} = h_0 \frac{\partial^2}{\partial \xi^2} + h_1 \xi \frac{\partial}{\partial \xi} + h_2 \frac{\partial}{\partial t} + h_3,$$

$$h_0(t) = \frac{1}{2} H_{pp}, \quad h_1(t) = \mathrm{i}(bH_{pp} + H_{pq}), \quad h_2 = \mathrm{i},$$

$$h_3(t) = \frac{\mathrm{i}}{2\mathrm{H}} \left\{ bHH_{pp} - \dot{\omega} - 2H_q H_p + \ddot{q}p + \frac{1}{\eta} \int_{s_1[q(t)]}^{s_2[q(t)]} L_\omega \dot{y} y \, \mathrm{d}s + \Gamma \right\},$$

$$\Gamma(t) = -\frac{4k[q(t)]k'[q(t)]\lambda[q(t)]}{p^5(t)} - p(t)\frac{\partial T_2}{\partial \varphi}[q(t), t]$$

$$- \frac{p(t)}{\eta(t)} T_3[s, q(t), t] y^2[s, q(t)] \Big|_{s_1[q(t)]}^{s_2[q(t)]}, \quad \eta(t) = \int_{s_1[q(t)]}^{s_2[q(t)]} y^2 \, \mathrm{d}s. \tag{3.34}$$

Equation (3.33) has a solution of polynomial form if and only if the function $b(t)$ satisfies the Riccati equation

$$\dot{b} + \frac{\partial^2 H}{\partial p^2} b^2 + 2\frac{\partial^2 H}{\partial p \partial q} b + \frac{\partial^2 H}{\partial q^2} = 0. \tag{3.35}$$

From the initial conditions (2.8) (3.10) and equation (3.12), it follows

$$b(0) = b^{\circ}. \tag{3.36}$$

Let $b(t)$ be the solution of equation (3.35) with the initial conditions (3.36). It can be proved [10] that if the functions $k(\varphi)$, $s_i(\varphi)$, $T_2(\varphi, t)$ are infinitely differentiable with respect to φ, and the stress $T_2(\varphi, t)$ is such that the radicand in the Hamiltonian function (3.20) is positive for any $t \geqslant 0$, then the inequality $\operatorname{Im} b^{\circ}(t) > 0$ implies the inequality $0 < \operatorname{Im} b(t) < +\infty$ at any finite interval $0 < t < \hat{t}$. So, the required condition (3.13) is fulfilled here.

Reverting to equation (3.33), upon taking into consideration the Riccati equation (3.35), one obtains the amplitude equation

$$D_{\xi t} P_0 = 0. \tag{3.37}$$

The solution of this equation is the polynomial

$$P_0(\xi, t; \alpha_j) = \sum_{k=0}^{M} A_k(t)\xi^k \tag{3.38}$$

of the Mth degree with coefficients

$$A_M(t) = \alpha_M \Psi_M(t), \quad A_{M-1}(t) = \alpha_{M-1}\Psi_{M-1}(t),$$

$$A_{M-r}(t) = \Psi_{M-r}(t)\left\{\alpha_{M-r} - (M-r+2)(M-r+1)\int_0^t \frac{h_0(t)A_{M-r+2}(t)}{h_2(t)\Psi_{M-r}(t)}\,\mathrm{d}t\right\},$$

$$\Psi_j(t) = \exp\left\{-\int_0^t \frac{jh_1(t)+h_3(t)}{h_2(t)}\,\mathrm{d}t\right\}, \quad r = 2, 3, \ldots, M; \quad j = 0, 1, \ldots, M, \tag{3.39}$$

where α_j are arbitrary complex numbers, which can be determined from the initial conditions (3.10).

A solution of equation (3.37) may be expressed by means of Hermite polynomials. Such presentation will be suitable in one special case to compare expansion (3.12) with a solution constructed by Tovstik [8] in the problem on free vibrations of an elastic cylindrical shell near the weakest generator.

Let $x = \vartheta(t)\xi$ be a new independent variable, where $\vartheta(t)$ is an unknown function. As a result, equation (3.37) is replaced by

$$\frac{\partial^2 P_0}{\partial x^2} + \frac{1}{\vartheta^2}\left(\frac{h_1}{h_0} + \frac{h_2}{h_0}\frac{\dot{\vartheta}}{\vartheta}\right)x\frac{\partial P_0}{\partial x} + \frac{h_2}{h_0\vartheta^2}\frac{\partial P_0}{\partial t} + \frac{h_3}{h_0\vartheta^2}P_0 = 0. \tag{3.40}$$

The function $\vartheta(t)$ may be found from the non-linear equation

$$\frac{1}{\vartheta^2}\left(\frac{h_1}{h_0} - \frac{h_2}{h_0}\frac{\dot{\vartheta}}{\vartheta}\right) = -2. \tag{3.41}$$

This equation has the following two branches of solutions:

$$\vartheta(t) = \pm\frac{\exp[-\int(h_1/h_2)\,\mathrm{d}t]}{\sqrt{\tilde{c} + 4\int(h_0/h_2)\exp[-2\int(h_1/h_2)\,\mathrm{d}t]\,\mathrm{d}t}}, \tag{3.42}$$

where $\tilde{c}$ is an unknown constant. Picking out both a sign and a constant $\tilde{c}$ is not essential here. For definiteness, in equation (3.42) the positive sign and $\tilde{c} = 0$ are assumed. Using equation (3.42), a solution of equation (3.40) may be sought in the form

$$P_0 = \chi(t)X(x). \tag{3.43}$$

Inserting equation (3.43) into equation (3.40) gives

$$\frac{1}{X}\left(\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}X}{\mathrm{d}x}\right) = -\frac{1}{\chi}\left(\frac{h_2}{h_0\vartheta^2}\frac{\mathrm{d}\chi}{\mathrm{d}t} + \frac{h_3}{h_0\vartheta^2}\chi\right). \tag{3.44}$$

Hence,

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}X}{\mathrm{d}x} - c^*X = 0, \tag{3.45a}$$

$$\frac{h}{h_0\vartheta^2}\frac{\mathrm{d}\chi}{\mathrm{d}t} + \frac{h_3}{h_0\vartheta^2}\chi - c^*\chi = 0. \tag{3.45b}$$

where c^* is an unknown constant. Equation (3.45a) has the solution in the form of Hermit polynomial $X = \aleph_j(x)$ of the jth degree, if $c^* = -2j$. Then, solving equation (3.45b), one obtains

$$\chi = \chi_j(t) = \frac{\{4\int(h_0/h_2)\exp[-2\int(h_1/h_2)\,\mathrm{d}t]\,\mathrm{d}t\}^{j/2}}{\exp[\int(h_3/h_2)\,\mathrm{d}t]}. \tag{3.46}$$

Thus, the function $P_0 = \chi_j(t)\aleph_j(x)$ satisfies equation (3.40). It is evident that the polynomial

$$P_0(\xi, t; \beta_j) = \sum_{j=0}^{M} \beta_j \chi_j(t) \aleph_j[\vartheta(t)\xi] \tag{3.47}$$

of the Mth degree is also the solution of the amplitude equation (3.37). Arbitrary constants β_j are found from the initial conditions (3.10).

3.4. HIGHER APPROXIMATIONS

To determine the correction $\varepsilon^{m/2}w_m$ in equation (3.12) for $m \geqslant 1$, one must consider responding boundary-value problem (3.14), (3.16) in the $(m+2)$nd approximation. The existence of a solution of these problems leads to the non-homogeneous differential equation $D_{\xi t}P_m = P_m^*$ for the polynomial $P_m(\xi, t)$, where $P_m^*(\xi,t)$ is some polynomial depending on the polynomials $P_{m-1}, \dots, P_0$. However, it should be noted that the above procedure for constructing the functions w_m is no longer valid for $m \geqslant 4$ because the correction introduced by the boundary-value problem into the general solution (3.12) at the sixth step is of the order ε^2 at the shell edges, which is the same as the errors of the original boundary conditions (2.6) and of the governing equations (2.4) as well.

The function $w = [w_0(s, \xi, t) + O(\varepsilon^{1/2})]\exp[\mathrm{i}\varepsilon^{-1}S(\xi, t, \varepsilon)]$ found from the first three approximations is the main term in the asymptotic expansion (3.12).

3.5. DETERMINATION OF THE CONSTANTS

Taking equation (3.19) into account, denote by $p^\pm, q^\pm, \omega^\pm, b^\pm, P_0^\pm, w_0^\pm, f_0^\pm$ the positive and negative branches of the functions found above, corresponding to the Hamiltonians H^+ and H^-. Let $\xi^\pm = \varepsilon^{-1/2}[\varphi - q^\pm(t)]$. Consider the functions

$$\tilde{W} = w^+ + w^-, \quad \tilde{\Phi} = f^+ + f^-, \tag{3.48}$$

$$w^\pm = [w_0^\pm + O(\varepsilon^{1/2})]\exp(\mathrm{i}\varepsilon^{-1}S^\pm), \quad f^\pm = [f_0^\pm + O(\varepsilon^{1/2})]\exp(\mathrm{i}\varepsilon^{-1}S^\pm),$$

$$w_0^\pm = P_0^\pm(\xi^\pm, t; \alpha_j^\pm)y[s, q^\pm(t)],$$

$$f_0^\pm = P_0^\pm(\xi^\pm, t; \alpha_j^\pm)\frac{k[q^\pm(t)]}{[p^\pm(t)]^4}\frac{\partial^2 y}{\partial s^2}[s, q^\pm(t)],$$

$$S^\pm = \int_0^t \omega^\pm(\tau)\,\mathrm{d}\tau + \varepsilon^{1/2}p^\pm(t)\xi^\pm + \tfrac{1}{2}\varepsilon b^\pm(t)\xi^{\pm 2}.$$

By the above construction, the function $\tilde{W}$, $\tilde{\Phi}$ satisfy equation (3.9) in the first three approximations. Solution (3.48) contains arbitrary constants $\alpha_j^{\pm}$ or $\beta_j^{\pm}$ (if the polynomials $P_0^{\pm}$ are evaluated by equation (3.47)) which may be determined from the initial conditions. The substitution of equations (3.48) into equations (3.10), where the eigenfunction $y(s, \varphi)$ should be expanded into the series.

$$y(s, \varphi) = y(s, 0) + \varepsilon^{1/2}\zeta \frac{\partial y}{\partial \varphi}(s, 0) + \frac{1}{2}\varepsilon\zeta^2 \frac{\partial^2 y}{\partial \varphi^2}(s, 0) + \cdots$$

with regard to the equalities $\xi^{\pm} \equiv \zeta$, $y(s, q^{\pm}) \equiv y(s, 0)$ at $t = 0$, yields the two equations

$$P_0^{\pm}(\zeta, 0; \alpha_j^{\pm}) = \frac{1}{2}\left[w_0^{\circ}(\zeta) \mp \frac{v_0^{\circ}(\zeta)}{H^{\circ}} \right]. \tag{3.49}$$

Here $w_0^{\circ}(\zeta)$, $v_0^{\circ}(\zeta)$ are the polynomials evaluated by equations (3.5) (it should be remembered that the subscript n has been omitted), and $H^{\circ} = H^{+}(a^{\circ}, 0, 0)$. The equality conditions of the coefficients in equations (3.49) for the same degree of ζ give

$$\alpha_j^{\pm} = \frac{1}{2}\left(c_{0j}^{\circ} \mp \frac{d_{0j}^{\circ}}{H^{\circ}} \right), \quad j = 0, 1, \ldots, M_0. \tag{3.50}$$

From equations (3.50), it follows that the polynomials $P_0^{\pm}$ have the same degree M_0 that the polynomials $w_0^{\circ}(\zeta)$, $v_0^{\circ}(\zeta)$.

If the polynomials $P_0^{\pm}(\xi^{\pm}, t; \beta_j^{\pm})$ are calculated by equation (3.47), then

$$\beta_j^{\pm} = \frac{1}{2^{j+1} j! \sqrt{\pi}\chi_j(0)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\zeta^2} \aleph_j[\vartheta(0)\zeta] \left[w_0^{\circ}(\zeta) \mp \frac{v_0^{\circ}(\zeta)}{H^{\circ}} \right] \mathrm{d}\zeta. \tag{3.51}$$

4. ANALYSIS AND EXAMPLES

Analysis of solution (3.48) shows that if $q^{\pm}(t) \neq 0$ identically at $t > 0$ then the initial nth wave packet (3.10) splits into the n^{+}th and n^{-}th packets moving in the opposite directions with the group velocities $v_g^{\pm} = \dot{q}^{\pm}(t)$, the width of the packets being of the order $\varepsilon^{1/2}/\mathrm{Im}\, b^{\pm}(t)$.

Remark 1. Solution (3.48) is correct in the asymptotic sense at some segment $0 \leqslant t \leqslant t'$, where

$$\mathrm{Im}\, b^{\pm}(t) > 0, \tag{4.1}$$

$$\omega^{\pm}, p^{\pm}, b^{\pm}, \dot{\omega}^{\pm}, \dot{p}^{\pm}, \dot{b}^{\pm}, \dot{q}^{\pm}, w_j^{\pm}, f_j^{\pm}, \partial w_j^{\pm}/\partial \tilde{x}, \partial f_j^{\pm}/\partial \tilde{x} = O(1) \text{ at } \varepsilon \to 0, \tag{4.2}$$

$$p^{\pm}(t) \sim 1, \tag{4.3}$$

where $\tilde{x}$ denotes any of the variables s, ξ, t.

As it was mentioned above, inequality (4.1) holds if $\mathrm{Im}\, b^{\circ} > 0$, relations (4.2) are necessary for series (3.12) to be asymptotic ones, and estimation (4.3), being more strong then the corresponding estimation (4.2) for $p^{\pm}$, is introduced to satisfy relations (4.2) for the frequencies $\omega^{\pm}$ (see equations (3.19) and (3.20)). Here the symbol $\sim$ means that two quantities are of the same order (see the definition in the Appendix A).

Remark 2. It may be noticed that in the case when the shell is closed in the circumferential direction, solution (3.48) is not periodic with respect to φ; and in the case of a cylindrical panel, it does not take into account the influence of the edges $\varphi = \varphi_1, \varphi_2$. Hence, functions (3.48) may be used for a computation as long as the wave packet centers $\varphi = q^{\pm}(t)$ are far from the lines $\varphi = \varphi_1, \varphi_2$.

Remark 3. From equation (3.20) it follows that solution (3.48) should be considered at some segment $0 \leqslant t \leqslant t_b \leqslant t'$, where the inequality

$$(p^{\pm})^4 + \frac{\lambda(q^{\pm})k^2(q^{\pm})}{(p^{\pm})^4} - T_2(q^{\pm}, t)(p^{\pm})^2 > 0 \tag{4.4}$$

holds. If the hoop stress $T_2(\varphi, t)$ is negative (i.e., expanding stress), then condition (4.4) is valid for any t. Otherwise, the function $T_2(\varphi, t)$ should satisfy some additional restriction. Consider the special case

$$T_2(\varphi, t) = \Lambda(t)\tau(\varphi), \tag{4.5}$$

where $\Lambda(t) > 0$ and $\tau(\varphi) > 0$ if only on some part of the shell surface. The function $\Lambda(t)$ may be interpreted here as a parameter of the shell loading. Let

$$F(p, \varphi) = \frac{p^2}{\tau(\varphi)} + \frac{\lambda(\varphi)k^2(\varphi)}{p^6\tau(\varphi)}, \quad \Lambda_b = \min_{\varphi, p} F(p, \varphi) = F(p_b, \varphi_b), \tag{4.6}$$

where $p_b = [3\lambda(\varphi_b)k^2(\varphi_b)]^{1/8}$ and φ_b are found from the equation

$$-\frac{3\lambda k^2\tau'}{\tau^2} + \left(\frac{\lambda k^2}{\tau}\right)' = 0. \tag{4.7}$$

One can prove that the inequality $\Lambda(t) < \Lambda_b$ is the sufficient condition for the realization of condition (4.4). It should be noticed that if k, s_i, τ are constants, then $T_2 = \Lambda_b$ is the classical buckling hoop stress [7, 8].

Example 1. Analyze solution (3.48) for the circular cylindrical shell, with a constant length generatrix, subjected to the internal or external pressure $Q_3^* = \Lambda\varepsilon^6 R^{-1} Eh$. Here $k = 1$, $s_1 = 0$, $s_2 = l$, $T_2 = \Lambda$ are constant magnitudes. In this simplest case

$$p^{\pm} = a^{\circ}, \quad q^{\pm}(t) = \pm H_p t, \quad \omega^{\pm} = \pm H_p a^{\circ} \mp H^{\circ}, \quad b^{\pm}(t) = \frac{b^{\circ}}{C^{\pm}(t)}, \tag{4.8}$$

where

$$H^{\circ} = \sqrt{(a^{\circ})^4 + \lambda(a^{\circ})^{-4} - \Lambda(a^{\circ})^2}, \qquad H_p = \frac{2(a^{\circ})^8 - 2\lambda - \Lambda(a^{\circ})^6}{(a^{\circ})^5 H^{\circ}},$$

$$H_{pp} = \frac{2[(a^{\circ})^{16} + 12\lambda(a^{\circ})^8 + 3\lambda^2]}{(a^{\circ})^{10}(H^{\circ})^3} - \frac{3(a^{\circ})^6[(a^{\circ})^5 + 5\lambda]\Lambda}{(a^{\circ})^{10}(H^{\circ})^3}, \quad C^{\pm}(t) = 1 \pm b^{\circ}H_{pp}t.$$

For example, let $M_0 = 2$. Then, by using equations (3.39), one obtains the polynomials

$$P_0^{\pm}(\xi^{\pm}, t; \alpha_j^{\pm}) = \frac{\alpha_2^{\pm}(\xi^{\pm})^2}{[C^{\pm}(t)]^{5/2}} + \frac{\alpha_1^{\pm}\xi^{\pm}}{[C^{\pm}(t)]^{3/2}} + \frac{\alpha_0^{\pm}}{[C^{\pm}(t)]^{1/2}} \pm \frac{i\alpha_2^{\pm} H_{pp} t}{[C^{\pm}(t)]^{3/2}} \tag{4.9}$$

with the constants $\alpha_2^{\pm}$ estimated by formulas (3.50). It may be seen that in this case, the wave numbers $p^{\pm}$, the group velocities $v_g^{\pm}$, and the frequencies $|\omega^{\pm}(t)|$ are constants for any $t \geqslant 0$, with the packet width growing and the amplitude decreasing in both packets. In other words, in the shell having constant geometric parameters and experiencing constant pressure, the $n^{\pm}$th packets become dissolved.

Study the properties of solution (3.48) when k, s_i, T_2 are functions.

4.1. FREE VIBRATIONS NEAR THE WEAKEST LINE

At first, let $T_2 = 0$, and $G = \lambda(\varphi)k^2(\varphi)$, moreover,

$$G'(0) = 0, \quad G''(0) > 0. \tag{4.10}$$

This particular case is the most interesting and important one, since the line $\varphi = 0$ is the weakest one here. For instance, in the circular cylinder with a variable generator, the longest generator will be the weakest one. In its vicinity the eigenmodes [8]

$$\tilde{W} = [\aleph_j(\varepsilon^{-1/2}\vartheta_w\varphi)y(s,0) + O(\varepsilon^{1/2})]\exp\{i\varepsilon^{-1}(\omega_w t + p_w\varphi + \tfrac{1}{2}b_w\varphi^2)\} \tag{4.11}$$

of low-frequency vibrations are localized. Here $\vartheta_w = (H_{qq}/H_{pp})^{1/4}$, and

$$\omega_w = \omega_w^{o} + \varepsilon\omega_w^{(j)} + O(\varepsilon^2) \tag{4.12}$$

is the fundamental frequency, where

$$\omega_w^{o} = H(p_w, q_w, 0), \quad \omega_w^{(j)} = (j + \tfrac{1}{2})\vartheta_w^2, \quad j = 0, 1, 2, \ldots \tag{4.13}$$

and the numbers $p_w = g^{1/8}(0)$, $q_w = 0$, and $b_w = i\sqrt{H_{qq}/H_{pp}}$ are the solutions of the system

$$H_p = 0, \quad H_q = 0 \tag{4.14}$$

and of the equation

$$H_{pp}b^2 + 2H_{pq}b + H_{qq} = 0 \tag{4.15}$$

respectively (here $H_{pq} = 0$ at $p = p_w$, $q = 0$, $t = 0$). Equations (4.14) and (4.15) are the degenerate stationary analogies of the Hamiltonian system (3.27) and the Riccati equation (3.35) respectively.

Now, let $a^o = p_w$, $b^o = b_w$, where a^o, b^o are the parameters from the initial conditions (2.7), (2.8). It is apparent that in this case the solutions of the Hamiltonian system (3.27) and the Riccati equation (3.35) are

$$p^{\pm}(t) \equiv p_w, \; q^{\pm}(t) \equiv 0, \; b^{\pm}(t) \equiv b_w$$

for any $t \geqslant 0$. The polynomials $P_0^{\pm}$ may be defined in accordance with equation (3.47) as follows

$$P_0^{\pm}(\zeta, t; \beta_j^{\pm}) = \sum_{j=0}^{M} \beta_j^{\pm} \chi_j(t) \aleph_j[\vartheta(t)\zeta]. \tag{4.16}$$

Here

$$\vartheta = \vartheta_w,\ \chi_j(t) = \vartheta^{-j} \exp[-\mathrm{i}(j+\tfrac{1}{2})\sqrt{H_{pp}H_{qq}}\,t], \tag{4.17}$$

where the functions H_{pp}, H_{qq} are calculated at $p = p_w$, $q = 0$, $t = 0$, and the coefficients $\beta_j^{\pm}$ are expressed by equation (3.51). The substitution of equations (4.16) and (4.17) into solution (3.48) gives the function

$$\tilde{W} = \left\{ \sum_{j=0}^{M_0} (\beta_j^{+} + \beta_j^{-})(H_{pp}/H_{qq})^{j/4} \aleph_j[\varepsilon^{-1/2}(H_{qq}/H_{pp})^{1/4}\varphi]\, y(s, 0) + O(\varepsilon^{1/2}) \right\}$$

$$\times \exp\{\mathrm{i}\varepsilon^{-1}[\omega_w t + p_w\varphi + \tfrac{1}{2}b_w\varphi^2]\}, \tag{4.18}$$

which is the superposition of the $M_0 + 1$ number of eigenmodes (4.11), where M_0 is the degree of the polynomials $w_0^{\circ}(\zeta)$, $v_0^{\circ}(\zeta)$ in equations (3.5).

Now, let G not depend on φ (i.e., all the geometrical parameters of the shell are constant) and $T_2 = \Lambda\tau(\varphi)$, where Λ is constant and $\tau'(0) = 0$, $\tau''(0) < 0$. In this case, the generatrix $\varphi = 0$, on which the hoop stresses $T_2(\varphi)$ are maximum, will be the weakest [19]. Similarly, one can show here that at $a^{\circ} = p_w$, $b^{\circ} = b_w$ solution (3.48) is reduced to the stationary one (4.18).

Thus, if $a^{\circ} = p_w$, $b^{\circ} = b_w$, then the initial nth wave packet (3.10) with the center on the weakest generatrix $\varphi = 0$ is not splitted into the n^+ and n^- wave packets. In this particular case, solution (3.48) defines free vibrations localized near the weakest, i.e., the stationary wave packet.

4.2. TRAVELLING WAVE PACKETS

Analysis of the Hamiltonian system (3.27) shows that in the common case, when $a^{\circ} \neq p_w$, solution (3.48) represents the n^+th and n^-th wave packets moving in the opposite directions. Its properties depend strongly on $k(\varphi)$, $s_i(\varphi)$, $T_2(\varphi, t)$ and the initial conditions. In some cases, it is possible to describe the behavior of the travelling wave packets.

Consider the circular cylinder with the constant generatrix length l ($k = 1$, $s_1 = 0$, $s_2 = l$) subjected to non-uniform stationary pressure, so that the hoop stresses T_2 are evaluated by equation (4.5), where Λ is constant, and the function $\tau(\varphi)$ satisfies the conditions

$$\tau'(\varphi) > 0 \text{ at } \varphi_1 < \varphi < 0,\ \tau'(0) = 0,\ \tau'(\varphi) < 0 \text{ at } 0 < \varphi < \varphi_2. \tag{4.19}$$

The center of the initial nth wave packet (3.10) is supposed to be situated on the weakest line $\varphi = 0$, where the stresses T_2 is maximum. Introduce the additional notations

$$v_g^{\circ} = 2(a^{\circ})^8 - \Lambda\tau(0)(a^{\circ})^6 - 2\lambda, \quad p_r = \sqrt[4]{\frac{\sqrt{(H^{\circ})^4 + 12\lambda} - (H^{\circ})^2}{2}}, \quad \underline{\tau} = \inf_{0<\varphi<\varphi_2} \tau(\varphi),$$

where $\inf_{0<\varphi<\varphi_2}\tau(\varphi)$ denotes the greatest lower bound of the function $\tau(\varphi)$ in the interval $0<\varphi<\varphi_2$, and study the behavior of the nth wave packets running in the direction of increasing a co-ordinate φ.

There are the two essentially different variants:

$$v_g^{\rm o} \neq 0, \quad \Lambda > 2(p_r^2 - \lambda p_r^{-6})\underline{\tau}^{-1}, \tag{4.20a}$$

$$v_g^{\rm o} \neq 0, \quad \Lambda \leqslant 2(p_r^2 - \lambda p_r^{-6})\underline{\tau}^{-1}, \tag{4.20b}$$

The case $v_g^{\rm o}=0$ is not considered here because that is equivalent to the condition $a^{\rm o}=p_w$ for free vibrations near the weakest line (see above).

If inequalities (4.20a) are valid, then the analysis of the Hamiltonian system (3.27) gives

$$\dot p^+ < 0, \quad v_g^+ = \dot q^+ > 0, \quad \dot v_g^+ < 0 \quad \text{at } v_g^{\rm o} > 0,$$

$$\dot p^- < 0, \quad v_g^- = \dot q^- > 0, \quad \dot v_g^- < 0 \quad \text{at } v_g^{\rm o} < 0.$$

These inequalities show that in this case the wave packets run in the direction of the hoop stress diminution with decreasing group velocities.

Now, let conditions (4.20b) hold. Then at $v_g^{\rm o}>0$, there exist such $t_r^+>0$ and $t_0^+>t_r^+$ that

$$\dot p^+ < 0, \quad v_g^+ > 0, \quad \dot v_g^+ < 0 \quad \text{for } 0 < t < t_r^+,$$

$$\dot p^+(t_r^+) < 0, \quad v_g^+(t_r^+) = 0, \quad \dot v_g^+(t_r^+) < 0,$$

$$\dot p^+ < 0, \quad v_g^+ < 0, \quad \dot v_g^+ < 0 \quad \text{for } t_r^+ < t < t_0^+$$

and if $v_g^{\rm o}<0$, then there are $t_r^->0$ and $t_0^->t_r^-$ such that

$$\dot p^- > 0, \quad v_g^- > 0, \quad \dot v_g^- < 0 \quad \text{for } 0 < t < t_r^-,$$

$$\dot p^-(t_r^-) > 0, \quad v_g^-(t_r^-) = 0, \quad \dot v_g^-(t_r^-) < 0,$$

$$\dot p^- > 0, \quad v_g^- < 0, \quad \dot v_g^- < 0 \quad \text{for } t_r^- < t < t_0^-,$$

where $q^\pm(t_0^\pm)=0$. In other words, if conditions (4.20b) are satisfied, then the $n^\pm$th packets are reflected at $t=t_r^\pm$ from the generators $\varphi=q_r^\pm$, which can be determined from the equation $H(p_r,q,0)=H^{\rm o}$. At time $t=t_0^\pm$, the centers of the $n^\pm$th packets reset $\varphi=0$. Further dynamics of the $n^\pm$th packets depends upon the properties of the function $\tau(\varphi)$ at the interval $\varphi_1<\varphi<0$. In particular, if the function $\tau(\varphi)$ is even in the domain $-\varphi_2<\varphi<\varphi_2$, then one can prove that the functions $p^\pm(t)$, $q^\pm(t)$, $v_g^\pm(t)$, $\omega^\pm(t)$ will be periodic with the period $2t_0^\pm$. This means that the $n^\pm$th packets will make reciprocating motion near the weakest generator, being reflected from the generators $\varphi=q_r^\pm$.

The Riccati equation (3.35) as well as the amplitude equation (3.37) do not allow to find out the properties of the functions $b^\pm(t)$, $P_0^\pm(\xi^\pm,t,\alpha_j^\pm)$ characterizing the width and amplitude of the running packets, respectively, under any assumptions with respect to the functions $\tau(\varphi)$ and $G(\varphi)$. To analyze in detail the influence of both the geometrical parameters $k(\varphi)$, $s_i(\varphi)$ and the hoop stresses $T_2(\varphi,t)$, it is necessary to apply to the numerical calculations.

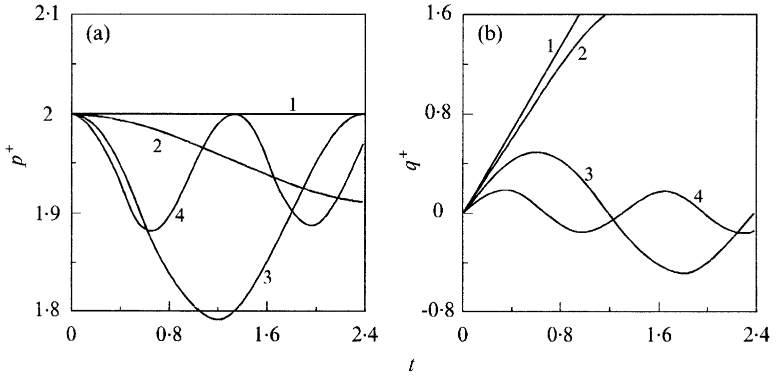

Figure 2. (a) Parameters p^+ and (b) centers q^+ of wave packets versus dimensionless time t in the circular cylindrical shell under the external "wind" pressure Q_3^*, for $l = 1$, $n = 1$, $a^o = 2$, $\Lambda = 2$ and various δ. 1, $\delta = 0$; 2, $\delta = 0{\cdot}05$; 3, $\delta = 0{\cdot}5$; 4, $\delta = 1$.

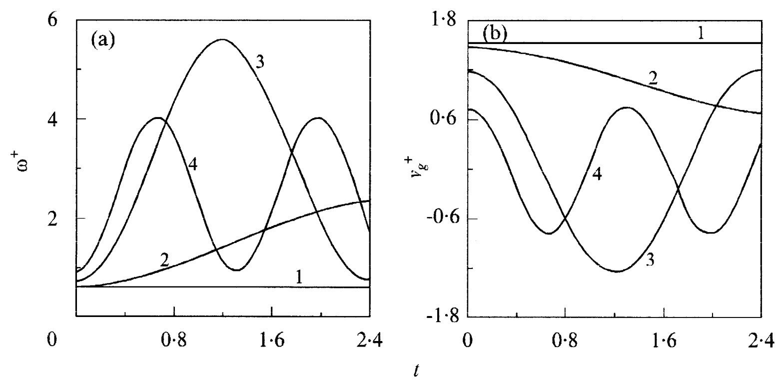

Figure 3. (a) Dimensionless frequencies ω^+ and (b) group velocities v_g^+ versus t in the circular cylindrical shell under the external "wind" pressure Q_3^*, for $l = 1$, $n = 1$, $a^o = 2$, $b^o = \mathrm{i}$, $\Lambda = 2$ and various δ. Key as Figure 2.

Example 2. Consider the joint-supported circular cylindrical shell, for which $k = 1$, $s_1 = 0$, $s_2 = l$, $\lambda = (\pi n/l)^4$, being under the non-uniform hoop stresses (4.5), where $\tau = 1 + \delta \cos \varphi$, $0 < \delta < 1$, $\Lambda > 0$. Such stresses are caused by the external "wind" normal pressure $Q_3^* = \varepsilon^6 R^{-1} E h \Lambda \tau(\varphi)$. Here the generator $\varphi = 0$ will be the weakest one. Calculations performed in reference [19] indicate that at $\Lambda = \Lambda_b = 4\pi 3^{-3/4}/[l(1 + \delta)]$ the shell buckles near the weakest generator. It is assumed that the center of the initial wave packet (3.10) coincides with this line.

Numerical computations for $l = 1$, $n = 1$, $a^o = 2$, $b^o = \mathrm{i}$, $w_0^o = 1$, $v_0^o = 0$, $\Lambda = 2$ and for various values of a parameter δ were performed (it is assumed here that $\Lambda = 2 < \Lambda_b$ for any δ). Figure 2 shows the solutions of the Hamiltonian system. It may be seen that, in the cases of uniform and non-uniform pressure with the low non-homogeneity ($\delta = 0$; 0·05), the 1^+st packet runs in the direction of pressure diminution without obstacles, whereas for $\delta = 0{\cdot}5$; 1 there are the effects of reflection of the 1^+st packet from the generators $\varphi = q_r^+ = 0{\cdot}16$ and $\varphi = q_r^+ = 0{\cdot}48$ respectively. Figure 3 demonstrates the manner in which the dimensionless momentary frequency ω^+ and the group velocity v_g^+ of the travelling packets of bending waves vary with the course of time, for the uniform pressure ($\delta = 0$) these magnitudes staying constant. In Figure 4, the parameter $\mathrm{Im}\, b^+$ and the maximum amplitude w_{max}^+ of

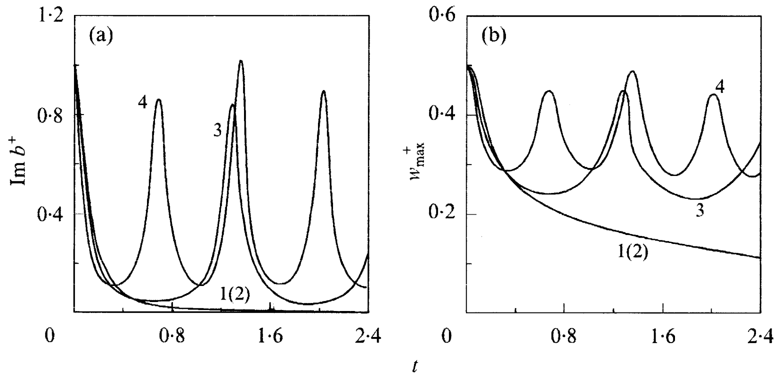

Figure 4. (a) Parameters $\mathrm{Im}\, b^+$ and (b) maximum amplitudes w_{max}^+ versus t in the circular cylindrical shell under the external "wind" pressure Q_3^*, for $l = 1$, $n = 1$, $a^o = 2$, $b^o = \mathrm{i}$, $w_0^o = 1$, $v_0^o = 0$, $\Lambda = 2$ and various δ. Key as Figure 2.

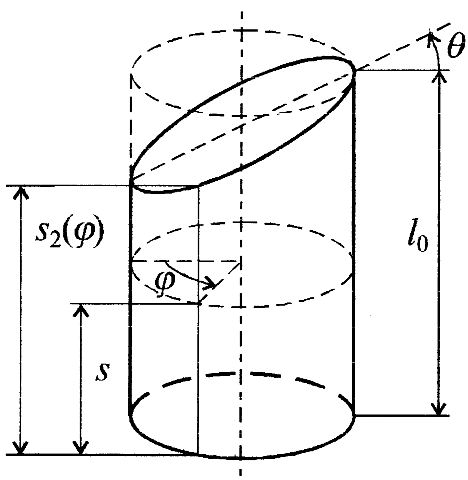

Figure 5. The circular cylindrical shell with the sloping edge.

bending waves in the 1^+st packet are plotted as functions of t. When comparing Figures 3 and 4, it may be concluded, that for small parameters δ characterizing the pressure non-homogeneity, the running packets become dissolved, but for large parameters δ the effects of reflecting packets are accompanied by focusing and growing amplitudes as well. Moreover, the larger the parameter δ is, the higher the power of focusing is and greater the magnitude of maximum amplitude becomes.

Example 3. Now, consider the joint-supported circular cylindrical shell with the sloping edge as shown in Figure 5. Here

$$k = 1,\ s_1 = 0,\ s_2(\varphi) = l_0 + (\cos\varphi - 1)\tan\theta$$

and the eigenvalue $\lambda(\varphi)$ and the eigenfunction $y(s, \varphi)$ of the boundary-value problem (3.1), (3.2a) are evaluated by equations (3.3). The case when the shell experiences the normal

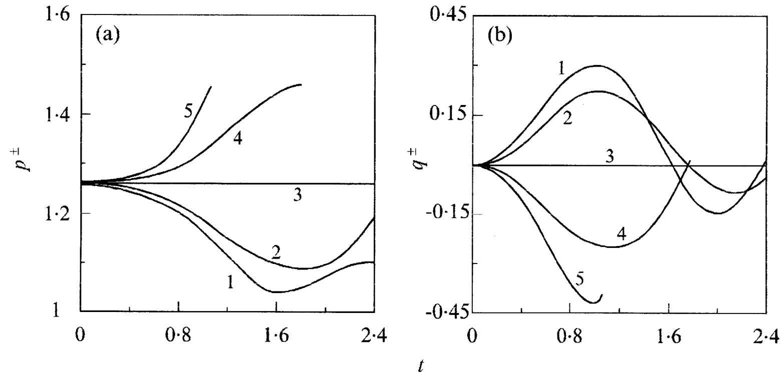

Figure 6. (a) Parameters $p^{\pm}$ and (b) centers $q^{\pm}$ versus dimensionless time t in the circular cylindrical shell with the oblique edge under the normal dynamic pressure Q_3^*, for $l_0 = 2$, $n = 1$, $\theta = 30°$ and various c_t. 1, $c_t = -3$; 2, $c_t = -2$; 3, $c_t = 0$; 4, $c_t = 1{\cdot}5$; 5, $c_t = 2{\cdot}5$.

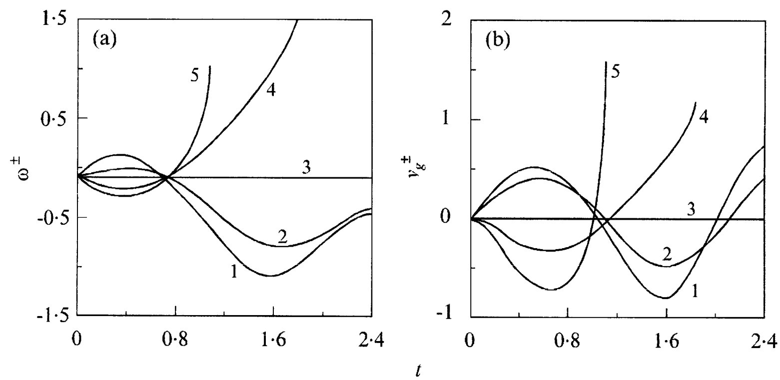

Figure 7. (a) Dimensionless frequencies $\omega^{\pm}$ and (b) group velocities $v_g^{\pm}$ versus t in the joint-supported circular cylindrical shell with the oblique edge under the normal dynamic pressure Q_3^*, for $l_0 = 2$, $n = 1$, $\theta = 30°$ and various c_t. Key as Figure 6.

dynamic pressure $Q_3^* = \varepsilon^6 R^{-1} E h c_t^* t^*$ is studied here, where $c_t^* = c_t / t_c^*$ $c_t \sim 1$ and t_c^* is the characteristic time introduced earlier. Then $T_2 = \Lambda(t) = c_t t$.

As mentioned above, the eigenmodes of low-frequency free vibrations of the shell under consideration (at $c_t = 0$) have form (4.11) and represent the stationary wave packet with the center on the weakest generator $\varphi = 0$ being the longest one. Let the initial wave packet (3.10) coincide with one of eigenmodes (4.11), where it is assumed

$$a^{\circ} = p_w = \sqrt{\pi n / l_0}, \quad b^{\circ} = b_w = \mathrm{i}\sqrt{H_{qq}/H_{pp}}, \quad w_0^{\circ} = \aleph_0 = 1, \; v_0^{\circ} = 0.$$

The problem is to analyze the influence of slowly growing external or internal pressure on eigenmode (4.11).

Graphs of the functions $p^{\pm}(t)$, $q^{\pm}(t)$, $\omega^{\pm}(t)$, $v_g^{\pm}(t)$, $\mathrm{Im}\, b^{\pm}(t)$, $w_{max}^{\pm}(t)$ are shown in Figures 6–8. Calculations were performed at $l_0 = 2$, $\theta = 30°$, $n = 1$ and for various values of a parameter c_t. In the case of external pressure ($c_t > 0$) computations were being conducted over the finite segment $0 < t < t_b$, where $\Lambda(t) < \Lambda_b$. Here $t_b \approx 1{\cdot}833$, $1{\cdot}020$ for $c_t = 1{\cdot}5$, $2{\cdot}5$ respectively. The figures show that growing pressure (both internal and external ones) splits

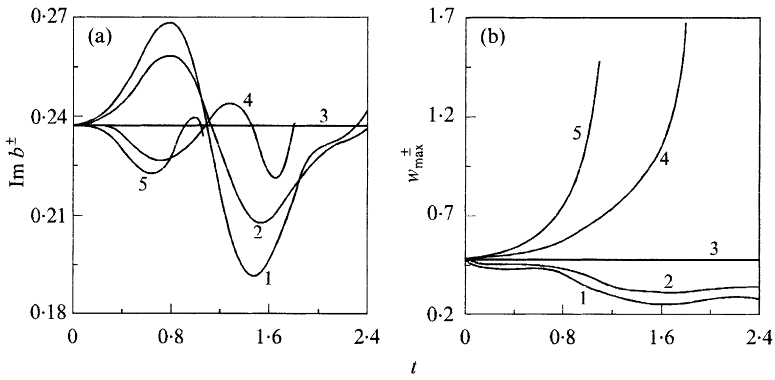

Figure 8. (a) Parameters $\text{Im}\, b^{\pm}$ and (b) maximum amplitudes $w^{\pm}_{max}$ versus t in the circular cylindrical shell with the oblique edge under the normal dynamic pressure Q_3^*, for $l_0 = 2$, $n = 1$, $\theta = 30°$, $w_0^o = 1$, $v_0^o = 0$ and various c_t. Key as Figure 6.

the initial wave packet coinciding with the eigenmode into the pair of the non-stationary wave packets moving in the opposite direction. However, the character of wave processes under the internal and external pressures are different. Under internal increasing pressure, one observes the multiple reflections of the packets from certain generators, these reflections being accompanied by focusing and slight growing wave amplitudes. When the pressure is external, the reflections of the wave packets are also possible, but the further behavior of the shell is distinguished by the very quick increase of the functions $|\omega^{\pm}(t)|$, $|v_g^{\pm}(t)|$, $|w^{\pm}_{max}(t)|$ at $t \to t_b$. It should be noted however that the unlimited growth of the foregoing functions contradicts conditions (4.2). Therefore, solution (3.48) should be considered in some interval $0 < t < t_b$ as long as the asymptotic correlations (4.2) are valid. Nevertheless, increasing amplitudes at $t \to t_b$ allows one to make an interesting assumption about the possibility of dynamic buckling of the shell under $\Lambda(t) < \Lambda_b$. But this is a non-linear problem which requires other approaches to solve.

5. CONCLUSIONS

By using the complex WKB method, the asymptotic solution of the initial-boundary-value problem for the equations, describing motion of a non-circular cylindrical shell with arbitrary edges, was constructed in the form of the superposition of packets of bending waves running in the circumferential direction. The properties of the obtained solution depend strongly on the geometrical parameters of the shell, the character of loading, and the initial conditions as well. In particular, if the shell has a weakest generator due to variable generator length or curvature, or pressure non-homogeneity, then the constructed solution permits one to investigate low-frequency free vibrations in a neighborhood of the weakest line. The qualitative analysis of the Hamiltonian system and the examples have revealed two interesting mechanical effects:

(1) The presence of the weakest generator on the shell surface may result in strong localization of the running packets of destructive bending waves, the packet reflections being accompanied by strong focusing and growing amplitudes.
(2) The initial local perturbations of the cylindrical shell having the weakest line and being under action of increasing external pressure may lead to very quick growing amplitudes of the running localized vibrations and, as a result, to dynamic buckling at the value of pressure which is less than the critical static pressure.

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APPENDIX

In this appendix the definition of the symbols O, o, $\sim$, and the asymptotic expansions used in the paper are given.

Let the functions $f(z)$ and $g(z)$ be defined on a set Ω of the complex numbers, $\mathbb{C}$, or the real numbers, $\mathbb{R}$, and let a be a point of accumulation of points of Ω.

Notation 1. We write $f(z) = O(g(z))$ as $z \to a$ if there exist a constant C and a neighborhood U of a such that $|f(z)| \leqslant C|g(z)|$ for any $z \in U$.

Notation 2. One write $f(z) = O(g(z))$ if there exists a constant C such that the inequality $|f(z)| \leqslant C|g(z)|$ holds for all $z \in \Omega$.

Notation 3. One writes $f(z) = o(g(z))$ as $z \to a$ if $\lim_{z \to a} f(z)/g(z) = 0$.

Notation $f(z) = O(g(z))$ means that the order of the function f is not larger than the order of the function g, and $f(z) = o(g(z))$ means that the order of f is less than the order of g as $z \to a$.

Notation 4. If $f(z) = O(g(z))$ and $g(z) = O(f(z))$ hold simultaneously as $z \to a$, we write $f(z) \sim g(z)$ as $z \to a$.

Operations on the symbols O and o and a large number of examples may be found, for example, in references [14, 15].

Consider a sequence of functions $u_m(z)$, $m = 0, 1, 2, \ldots$, defined on Ω and let a be a point of accumulation of Ω.

Definition 1. The sequence $u_m(z)$ is said to be *asymptotic* as $z \to a$, if for any integer $m \geqslant 0$,

$$u_{m+1}(z) = o(u_m(z)), \quad \text{as } z \to a.$$

For example, the sequence $u_m(z) = Z(z)(z - a)^m$ as $z \to a$, where $Z(z)$ is an arbitrary function in Ω, is the asymptotic one. Similar sequence appears in equation (3.12a). Indeed, the sequence $u_m(\varepsilon) = \varepsilon^{m/2} \exp[i\varepsilon^{-1} S(\xi, t, \varepsilon)]$ is the asymptotic as $\varepsilon \to 0$ for any fixed ξ, t.

Definition 2. Let the function $f(z)$ be defined on Ω and the sequence $u_m(z)$ be asymptotic as $z \to a$, then the series

$$f(z) \cong \sum_{m=0}^{\infty} a_m u_m(z) \quad \text{as } z \to a$$

is called an *asymptotic expansion of $f(z)$ in the Poincaré sense* by means of the asymptotic sequence $u_m(z)$ if there are constants a_m such that for any integer $M \geqslant 0$

$$f(z) - \sum_{m=0}^{M} a_m u_m(z) = o(u_M(z)) \quad \text{as } z \to a$$

Note that an asymptotic series may diverge. Basic properties of asymptotic series and operations on them are discussed in books [14, 15].